Solution to Problem Set 6

1. Let g be the polynomial defined by

$$g(x) \coloneqq a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots + \frac{a_{n-1}}{n}x^n + \frac{a_n}{n+1}x^{n+1}.$$

Then g is differentiable on $\mathbb{R}$ and $g' = f$, and also

$$g(0) = a_0 \cdot 0 + \frac{a_1}{2} \cdot 0^2 + \frac{a_2}{3} \cdot 0^3 + \cdots + \frac{a_{n-1}}{n} \cdot 0^n + \frac{a_n}{n+1} \cdot 0^{n+1} = 0,$$

$$g(1) = a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_{n-1}}{n} + \frac{a_n}{n+1} = 0.$$

Now $g(0) = g(1)$, so applying Rolle's Theorem to g, we see that there exists $c \in (0,1)$ such that

$$g'(c) = 0,$$

i.e. $f(c) = 0$. Therefore f has a root $c \in (0,1)$. ■

2. (a) Consider the function $f(x) = \cos x$, which is differentiable on $\mathbb{R}$ and

$$f'(x) = -\sin x.$$

Given any real numbers a and b, we consider the following cases:

(i) If $a = b$, then $|\cos a - \cos b| = 0 = |a - b|$, so the inequality holds trivially.

(ii) If $a \neq b$, then applying Mean Value Theorem to the function f we have

$$\frac{\cos a - \cos b}{a - b} = -\sin c$$

for some c between a and b. Since we always have $|-\sin c| \leq 1$, this implies that

$$\left|\frac{\cos a - \cos b}{a - b}\right| = |-\sin c| \leq 1,$$

so we still have $|\cos a - \cos b| \leq |a - b|$. ■

(b) Consider the function $g(x) = \tan x$, which is differentiable on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and

$$g'(x) = \sec^2 x.$$

Given any real numbers $a, b \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we consider the following cases:

(i) If $a = b$, then $|\tan a - \tan b| = 0 = |a - b|$, so the inequality holds trivially.

(ii) If $a \neq b$, then applying Mean Value Theorem to the function g we have

$$\frac{\tan a - \tan b}{a - b} = \sec^2 c$$

for some c between a and b. Since we always have $\sec^2 c \geq 1$, this implies that

$$\left|\frac{\tan a - \tan b}{a - b}\right| = |\sec^2 c| \geq 1,$$

so we still have $|\tan a - \tan b| \geq |a - b|$. ■

(c) Consider the function $h(x) = \ln(1+x)$, which is differentiable on $(-1, +\infty)$. Its derivative is

$$h'(x) = \frac{1}{1+x}.$$

Given any real number $t > -1$, we consider the following two cases:

(i) If $t = 0$, then $\ln(1+t) = 0 = \frac{t}{1+t}$, so the inequality holds trivially.

(ii) If $t \neq 0$, then applying Mean Value Theorem to the function h we have

$$\frac{\ln(1+t) - \ln(1+0)}{t-0} = \frac{1}{1+c}$$

for some c between 0 and t. In case $t > 0$, we have $0 < c < t$ and so

$$\frac{\ln(1+t)}{t} = \frac{\ln(1+t) - \ln(1+0)}{t-0} = \frac{1}{1+c} > \frac{1}{1+t}.$$

In case $-1 < t < 0$, we have $-1 < t < c < 0$ and so

$$\frac{\ln(1+t)}{t} = \frac{\ln(1+t) - \ln(1+0)}{t-0} = \frac{1}{1+c} < \frac{1}{1+t}.$$

In both cases we get $\ln(1+t) > \frac{t}{1+t}$, so the inequality still holds. ■

3. Let $f(t)$ be the distance (in km) that the runner travelled after t hours. We may assume that f is differentiable and its derivative f' is continuous. Now we have

$$f(0) = 0, \qquad f\left(\frac{1}{2}\right) = 10, \qquad f'(0) = 0 \qquad \text{and} \qquad f'\left(\frac{1}{2}\right) = 0.$$

Applying Mean Value Theorem to f, we see that there exists $c \in \left(0, \frac{1}{2}\right)$ such that

$$f'(c) = \frac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0} = \frac{10-0}{\frac{1}{2} - 0} = 20.$$

Since $f'(0) = 0 < 19$, $f'(c) = 20 > 19$ and $f'\left(\frac{1}{2}\right) = 0 < 19$, applying Intermediate Value Theorem to the continuous function f', we see that there exists $c_1 \in (0, c)$ and $c_2 \in \left(c, \frac{1}{2}\right)$ such that

$$f'(c_1) = 19 \qquad \text{and} \qquad f'(c_2) = 19.$$

In other words, he must have been running at a speed of exactly $19\ \mathrm{km/h}$ at least twice during the race.

4. Since f is twice differentiable on $\mathbb{R}$, it follows that the function g is continuous on $[a, b]$ and differentiable on (a, b). Its derivative is given by

$$\begin{aligned} g'(x) &= f'(x) - \big(f''(x)(x-b) + f'(x)\big) - \frac{f(a) - f(b) - f'(a)(a-b)}{(a-b)^2} \cdot 2(x-b) \\ &= 2(b-x)\left[\frac{f''(x)}{2} - \frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}\right] \end{aligned}$$

for every $x \in (a, b)$.

Now we have

$$g(a) = f(a) - f(b) - f'(a)(a-b) - \frac{f(a) - f(b) - f'(a)(a-b)}{(a-b)^2}(a-b)^2 = 0$$

and

$$g(b) = f(b) - f(b) - f'(b)(b-b) - \frac{f(a) - f(b) - f'(a)(a-b)}{(a-b)^2}(b-b)^2 = 0,$$

so $g(a) = g(b)$. Applying Rolle's Theorem on this function g, there exists $c \in (a,b)$ such that $g'(c) = 0$, i.e.

$$2(b-c)\left[\frac{f''(c)}{2} - \frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}\right] = 0.$$

Since $b \neq c$, this implies $\frac{f''(c)}{2} = \frac{f(b)-f(a)-f'(a)(b-a)}{(b-a)^2}$, and so $f(b) = f(a) + f'(a)(b-a) + \frac{f''(c)}{2}(b-a)^2$. ∎

5. (a) The statement is true.

We first note that there exists $t \in (2,5)$ such that $f(t) < 0$. To see this, we suppose on the contrary that $f(t) \geq 0$ for every $t \in (2,5)$. Then

$$f'_+(2) = \lim_{t \to 2^+} \frac{f(t) - f(2)}{t-2} \geq 0,$$

which contradicts with $f'(2) = -1$.

Next, since f is continuous, $f(t) < 0$ and $f(5) = 2 > 0$, applying Intermediate Value Theorem to f, there exists a number $c \in (t,5)$ such that $f(c) = 0$.

Since $f(c) = f(2)$ but $c \neq 2$, we see that f is not one-to-one. So f does not have an inverse.

(b) The statement is true.

Applying Mean Value Theorem to g, there exists $c \in (0,2)$ such that

$$g'(c) = \frac{g(2) - g(0)}{2-0} = \frac{0-(-2)}{2-0} = 1.$$

Remark: Here we cannot apply Intermediate Value Theorem to g', because g' does not have to be continuous.

6. (a) The derivative of f is given by

$$f'(x) = (2x)(x+4)^{\frac{2}{3}} + (x^2)\left[\frac{2}{3}(x+4)^{-\frac{1}{3}}(1)\right] = \frac{8x(x+3)}{3(x+4)^{\frac{1}{3}}}$$

for every $x \in \mathbb{R} \setminus \{-4\}$. Since $f'(x) = 0$ only if $x = 0$ or $x = -3$, the critical numbers of f are -4, -3 and 0.

(i) For $x < -4$, we have $f'(x) < 0$. So f is decreasing on $(-\infty, -4)$.

(ii) For $-4 < x < -3$, we have $f'(x) > 0$. So f is increasing on $(-4, -3)$.

(iii) For $-3 < x < 0$, we have $f'(x) < 0$. So f is decreasing on $(-3, 0)$.

(iv) For $x > 0$, we have $f'(x) > 0$. So f is increasing on $(0, +\infty)$.

We have the following table:

x	$(-\infty,-4)$	-4	$(-4,-3)$	-3	$(-3,0)$	0	$(0,+\infty)$
$f'(x)$	$-$	undef.	$+$	0	$-$	0	$+$
$f(x)$	$\searrow$	min.	$\nearrow$	max.	$\searrow$	min.	$\nearrow$

So f has a local maximum point $(-3,9)$ and local minimum points $(-4,0)$ and $(0,0)$.

(b) The derivative of g is given by

$$g'(x)=\frac{(3x^2+9)(x^2+1)-(x^3+9x)(2x)}{(x^2+1)^2}=\frac{(x^2-3)^2}{(x^2+1)^2}$$

for every $x\in\mathbb{R}$. Since $g'(x)\geq 0$ for every $x\in\mathbb{R}$, the sign of g' never changes, so g has no local extremum points. [g has critical numbers $\sqrt{3}$ and $-\sqrt{3}$ though.]

(c) Noting that

$$h(x)=\begin{cases}\dfrac{x}{(x+1)^2} & \text{if } x\geq 0\\ -\dfrac{x}{(x+1)^2} & \text{if } x<0 \text{ and } x\neq -1\end{cases},$$

we see that

(i) For $x>0$, we have

$$h'(x)=\frac{(1)(x+1)^2-(x)[2(x+1)]}{(x+1)^4}=\frac{1-x}{(x+1)^3}.$$

(ii) For $x<0$ and $x\neq -1$, we have

$$h'(x)=-\frac{(1)(x+1)^2-(x)[2(x+1)]}{(x+1)^4}=\frac{x-1}{(x+1)^3}.$$

(iii) For $x=0$, we have

$$h'_-(0)=\lim_{x\to 0^-}\frac{-\dfrac{x}{(x+1)^2}-0}{x}=-1 \qquad \text{and} \qquad h'_+(0)=\lim_{x\to 0^+}\frac{\dfrac{x}{(x+1)^2}-0}{x}=1,$$

so $h'(0)$ does not exist.

Now $h'(x)=0$ only if $x=1$, so the critical numbers of h are 0 and 1.

-1 is outside the domain of h.

(i) For $x<-1$, we have $h'(x)>0$. So h is increasing on $(-\infty,-1)$.

(ii) For $-1<x<0$, we have $h'(x)<0$. So h is decreasing on $(-1,0)$.

(iii) For $0<x<1$, we have $h'(x)>0$. So h is increasing on $(0,1)$.

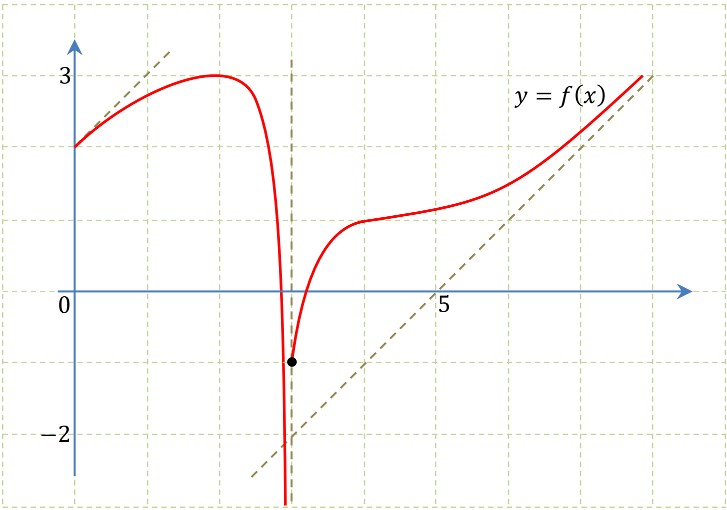
(iv) For $x>1$, we have $h'(x)<0$. So h is decreasing on $(1,+\infty)$.

We have the following table:

x	$(-\infty,-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,+\infty)$
$h'(x)$	$+$	undef.	$-$	undef.	$+$	0	$-$
$h(x)$	$\nearrow$	undef.	$\searrow$	min.	$\nearrow$	max.	$\searrow$

So h has a local maximum point $\left(1,\frac{1}{4}\right)$ and a local minimum point $(0,0)$.

7.



8. (a) The domain of f is $\mathbb{R} \setminus \{0\}$, i.e. $(-\infty, 0) \cup (0, +\infty)$. The derivatives of f are given by

$$f'(x) = e^{\frac{1}{x}} + xe^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) = \left(1 - \frac{1}{x}\right)e^{\frac{1}{x}}$$

$$f''(x) = \left(\frac{1}{x^2}\right)e^{\frac{1}{x}} + \left(1 - \frac{1}{x}\right)e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) = \frac{1}{x^3}e^{\frac{1}{x}}$$

for every $x \neq 0$. We have $f'(x) = 0$ only if $x = 1$, and $f''(x)$ is always non-zero.

We construct the following table:

x	$(-\infty, 0)$	0	$(0, 1)$	1	$(1, +\infty)$
$f'(x)$	$+$	undef.	$-$	0	$+$
$f''(x)$	$-$	undef.	$+$	$+$	$+$
$f(x)$	↗	undef.	↘	min.	↗

So f has a local minimum point $(1, e)$, no local maximum points and no points of inflection.

Now we look for asymptotes of the graph of f.

(i) Since

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} xe^{\frac{1}{x}} = \lim_{x\to0^-} x \cdot \lim_{x\to0^-} e^{\frac{1}{x}} = 0 \quad \text{and} \quad \lim_{x\to0^+} f(x) = \lim_{x\to0^+} xe^{\frac{1}{x}} = \lim_{t\to+\infty} \frac{e^t}{t} = +\infty,$$

the line $x = 0$ is a vertical asymptote of the graph of f.

(ii) Since

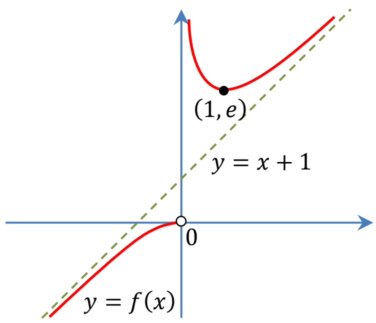
$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} e^{\frac{1}{x}} = e^0 = 1$$

$$\lim_{x\to\pm\infty} [f(x) - x] = \lim_{x\to\pm\infty} \left(xe^{\frac{1}{x}} - x\right) = \lim_{t\to0} \frac{e^t - 1}{t} = 1,$$

the line $y = x + 1$ is a slant asymptote of the graph of f.

Clearly the graph of f has no x- and y-intercepts.

A sketch of the graph of f is shown on the right.



(b) The domain of f is $\mathbb{R} \setminus \{6\}$, i.e. $(-\infty, 6) \cup (6, +\infty)$. Let's skip the routine computational details of the derivatives of f. They are given by

$$f'(x) = \frac{(x+8)(x+1)(x-27)}{(x-6)^3} \quad \text{and} \quad f''(x) = \frac{686(x+3)}{(x-6)^4}$$

for every $x \neq 6$. We have $f'(x) = 0$ only if $x = -8$, -1 or 27, while $f''(x) = 0$ only if $x = -3$.

We construct the following table:

x	$(-\infty,-8)$	-8	$(-8,-3)$	-3	$(-3,-1)$	-1	$(-1,6)$	6	$(6,27)$	27	$(27,+\infty)$
$f'(x)$	$+$	0	$-$	$-$	$-$	0	$+$	undef.	$-$	0	$+$
$f''(x)$	$-$	$-$	$-$	0	$+$	$+$	$+$	undef.	$+$	$+$	$+$
$f(x)$	$\nearrow$	max.	$\searrow$	infl.	$\searrow$	min.	$\nearrow$	undef.	$\searrow$	min.	$\nearrow$

f has local minimum points $(-1, 0)$ and $\left(27, \frac{224}{3}\right)$, a local maximum point $\left(-8, \frac{7}{4}\right)$ and a point of inflection $\left(-3, \frac{16}{27}\right)$.

Now we look for asymptotes of the graph of f.

(i) Since

$$\lim_{x \to 6} f(x) = +\infty,$$

the line $x = 6$ is a vertical asymptote of the graph of f.

(ii) Since

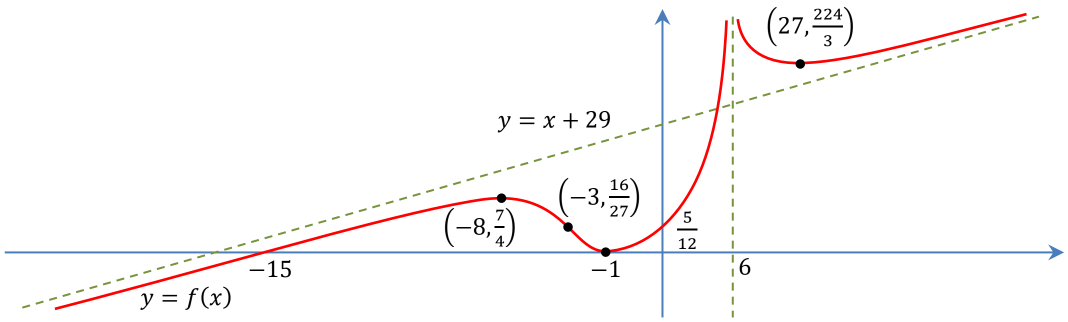
$$\lim_{x \to \pm\infty} (f(x) - x) = \lim_{x \to \pm\infty} \left(\frac{(x+15)(x+1)^2}{(x-6)^2} - x \right) = \lim_{x \to \pm\infty} \frac{29x^2 - 5x + 15}{(x-6)^2}$$

$$= \lim_{x \to \pm\infty} \frac{29 - \frac{5}{x} + \frac{15}{x^2}}{\left(1 - \frac{6}{x}\right)^2} = \frac{29 - 0 + 0}{(1-0)^2} = 29,$$

the line $y = x + 29$ is a slant asymptote of the graph of f.

The graph of f has x-intercepts -15 and -1, and has a y-intercept $\frac{5}{12}$.

The following shows a sketch of the graph of f.



(c) The domain of f is $\mathbb{R}\setminus\{-1\}$. Since $f(x)=\begin{cases}\frac{x^3-3x^2}{x+1} & \text{if } x>3 \text{ or } x<-1\\ -\frac{x^3-3x^2}{x+1} & \text{if } -1<x\le 3\end{cases}$, the derivatives of f are

$$f'(x)=\begin{cases}\dfrac{2x(x-\sqrt{3})(x+\sqrt{3})}{(x+1)^2} & \text{if } x>3 \text{ or } x<-1\\ -\dfrac{2x(x-\sqrt{3})(x+\sqrt{3})}{(x+1)^2} & \text{if } -1<x<3\end{cases},\qquad f''(x)=\begin{cases}2-\dfrac{8}{(x+1)^3} & \text{if } x>3 \text{ or } x<-1\\ \dfrac{8}{(x+1)^3}-2 & \text{if } -1<x<3\end{cases}.$$

It can be checked from definition that $f'_-(3)=-\frac{9}{4}$ and $f'_+(3)=\frac{9}{4}$, so $f'(3)$ does not exist.

We have $f'(x)=0$ only if $x=-\sqrt{3}$, $x=0$ or $x=\sqrt{3}$, while $f''(x)=0$ only if $x=\sqrt[3]{4}-1$.

Thus we have the following table.

x	$(-\infty,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},-1)$	-1	$(-1,0)$	0	$(0,\sqrt[3]{4}-1)$
$f'(x)$	$-$	0	$+$	undef.	$-$	0	$+$
$f''(x)$	$+$	$+$	$+$	undef.	$+$	$+$	$+$
$f(x)$	↘	min.	↗	undef.	↘	min.	↗

x	$\sqrt[3]{4}-1$	$(\sqrt[3]{4}-1,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},3)$	3	$(3,+\infty)$
$f'(x)$	$+$	$+$	0	$-$	undef.	$+$
$f''(x)$	0	$-$	$-$	$-$	undef.	$+$
$f(x)$	infl.	↗	max.	↘	min. corner	↗

h has one local maximum point $(\sqrt{3},6\sqrt{3}-9)$, three local minimum points $(-\sqrt{3},9+6\sqrt{3})$, $(0,0)$ and $(3,0)$, and two points of inflection $(\sqrt[3]{4}-1,6\sqrt[3]{4}-9)$ and $(3,0)$.

Next we look for asymptotes of the graph of f.

(i) Since $\lim\limits_{x\to-1}f(x)=+\infty$, the line $x=-1$ is a vertical asymptote of the graph of f.

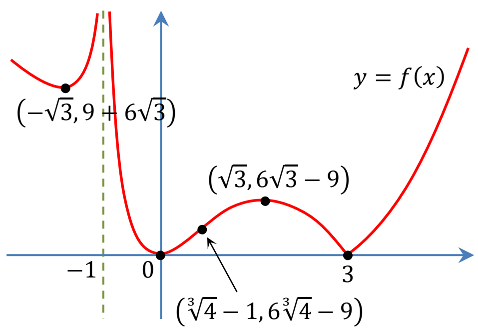
(ii) Since

$$\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}x\left|\frac{x-3}{x+1}\right|=\pm\infty,$$

the graph of f has no horizontal or slant asymptotes.

The graph of f has x-intercepts 0 and 3, and has a y-intercept 0.

A sketch of the graph of f is shown below.



(d) The domain of f is $\mathbb{R} \setminus (0,1] = (-\infty, 0] \cup (1, +\infty)$. Since $f(x) = \begin{cases} (x-2)\sqrt{\frac{x}{x-1}} & \text{if } x \geq 2 \\ (2-x)\sqrt{\frac{x}{x-1}} & \text{if } x \leq 0 \text{ or } 1 < x < 2 \end{cases}$,

the derivatives of f are given by

$$f'(x) = \begin{cases} \dfrac{(4x-3)^2+7}{16\sqrt{x(x-1)^3}} & \text{if } x > 2 \\ -\dfrac{(4x-3)^2+7}{16\sqrt{x(x-1)^3}} & \text{if } x < 0 \text{ or } x \in (1,2) \end{cases}, \quad f''(x) = \begin{cases} \dfrac{2-5x}{4x(x-1)\sqrt{x(x-1)^3}} & \text{if } x > 2 \\ \dfrac{5x-2}{4x(x-1)\sqrt{x(x-1)^3}} & \text{if } x < 0 \text{ or } x \in (1,2) \end{cases}.$$

Since $f'_-(2) = -\sqrt{2}$ and $f'_+(2) = \sqrt{2}$, f is not differentiable at 2.

Note that f' and f'' are always non-zero, and also $\lim\limits_{x \to 0^-} p'(x) = -\infty$. Thus we have the following table.

x	$x < 0$	$x = 0$	$0 < x \leq 1$	$1 < x < 2$	$x = 2$	$x > 2$
$f'(x)$	$-$	undef.	undef.	$-$	undef.	$+$
$f''(x)$	$-$	undef.	undef.	$+$	undef.	$-$
$f(x)$	$\searrow$	min.	undef.	$\searrow$	min. corner	$\nearrow$

f has local minimum points $(0,0)$ and $(2,0)$, no local maximum points and a point of inflection $(2,0)$.

Next we look for asymptotes of the graph of f.

(i) Since $\lim\limits_{x \to 1^+} f(x) = +\infty$, the line $x = 1$ is a vertical asymptote of the graph of f.

(ii) We have

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \left(1 - \frac{2}{x}\right)\sqrt{\frac{1}{1-\frac{1}{x}}} = 1 \qquad \text{and} \qquad \lim_{x \to -\infty} \frac{f(x)}{x} = \lim_{x \to -\infty} \left(\frac{2}{x} - 1\right)\sqrt{\frac{1}{1-\frac{1}{x}}} = -1,$$

and

$$\lim_{x \to +\infty} [f(x) - x] = \lim_{x \to +\infty} \left[(x-2)\sqrt{\frac{x}{x-1}} - x\right] = \lim_{x \to +\infty} \frac{(x-2)^2 \frac{x}{x-1} - x^2}{(x-2)\sqrt{\frac{x}{x-1}} + x}$$

$$= \lim_{x \to +\infty} \frac{-3x^2 + 4x}{(x-1)(x-2)\sqrt{\frac{x}{x-1}} + (x-1)x}$$

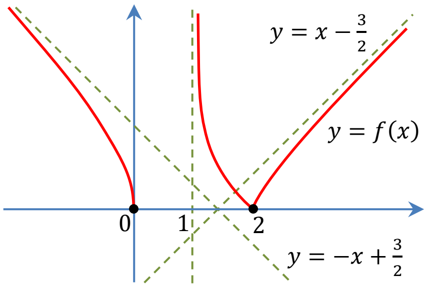
$$= \lim_{x \to +\infty} \frac{-3 + \frac{4}{x}}{\left(1 - \frac{1}{x}\right)\left(1 - \frac{2}{x}\right)\sqrt{\frac{1}{1 - 1/x}} + \left(1 - \frac{1}{x}\right)} = -\frac{3}{2},$$

and similarly

$$\lim_{x \to -\infty} [f(x) - (-x)] = \frac{3}{2}.$$

Therefore the lines $y = x - \frac{3}{2}$ and $y = -x + \frac{3}{2}$ are slant asymptotes of the graph of f.

The graph of f has x-intercepts 0 and 2, and has a y-intercept 0. A sketch of the graph of f is shown on the right.



9. (a) The domain of f is $\mathbb{R} \setminus \{1\}$, i.e. $(-\infty, 1) \cup (1, +\infty)$. The derivatives of f are given by

$$f'(x) = 2x + \frac{8}{(x-1)^2} = \frac{(x+1)[(2x-3)^2+7]}{2(x-1)^2}$$

$$f''(x) = 2 - \frac{16}{(x-1)^3}$$

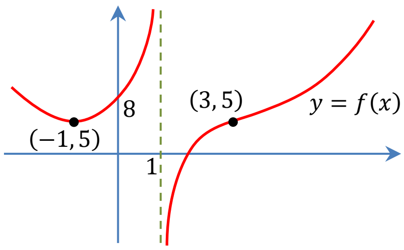
for every $x \neq 1$. So $f'(x) = 0$ only if $x = -1$ and $f''(x) = 0$ only if $x = 3$.

x	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, 3)$	3	$(3, +\infty)$
$f'(x)$	$-$	0	$+$	undef.	$+$	$+$	$+$
$f''(x)$	$+$	$+$	$+$	undef.	$-$	0	$+$
$f(x)$	↘	min.	↗	undef.	↗	infl.	↗

f has a local minimum point $(-1, 5)$, no local maximum points, and a point of inflection $(3, 5)$.

The graph of f has a vertical asymptote $x = 1$, and no horizontal or slant asymptotes. It has a y-intercept 8.

A sketch of the graph of f is shown on the right.



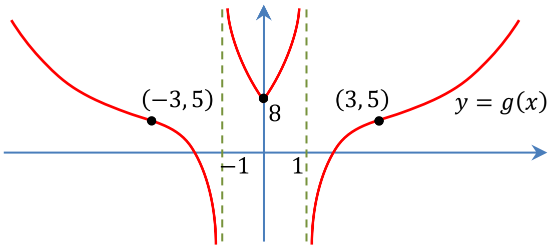
(b) (i) We compute

$$g'_-(0) = \lim_{x\to0^-} \frac{g(x)-g(0)}{x} = \lim_{x\to0^-} \frac{f(-x)-f(0)}{x} = \lim_{x\to0^-} \frac{(-x)^2 - \frac{8}{-x-1} - 8}{x} \lim_{x\to0^-} \frac{x^2+x-8}{x+1} = -8$$

$$g'_+(0) = \lim_{x\to0^+} \frac{g(x)-g(0)}{x} = \lim_{x\to0^+} \frac{f(x)-f(0)}{x} = \lim_{x\to0^+} \frac{x^2 - \frac{8}{x-1} - 8}{x} \lim_{x\to0^+} \frac{x^2-x-8}{x-1} = 8.$$

Since $g'_-(0) \neq g'_+(0)$, g is not differentiable at 0.

(ii) A sketch of the graph of g is shown below.



10. (a) It is clear that $g(0) = 0 = h(0)$. Next we compute the derivatives of g and h. We have

$$g'(x) = \frac{(2-4x)(2x+1)-(2x-2x^2)(2)}{(2x+1)^2} = \frac{3}{(2x+1)^2} - 1$$

for every $x \neq -\frac{1}{2}$, and

$$h'(x) = \frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}} \frac{(2)(1+x^2)-(2x)(2x)}{(1+x^2)^2} = \frac{2}{1+x^2}$$

for every $-1 < x < 1$. Therefore $g'(0) = 2 = h'(0)$. ■

(b) The domain of f is $\left(-\infty,-\frac{1}{2}\right)\cup\left(-\frac{1}{2},1\right)$. For $x\neq 0$, the derivatives of f are given by

$$f'(x)=\begin{cases} g'(x) & \text{if } x<0 \text{ and } x\neq -\dfrac{1}{2} \\ h'(x) & \text{if } 0<x<1 \end{cases} = \begin{cases} \dfrac{3}{(2x+1)^2}-1 & \text{if } x<0 \text{ and } x\neq -\dfrac{1}{2} \\ \dfrac{2}{1+x^2} & \text{if } 0<x<1 \end{cases}$$

$$f''(x)=\begin{cases} -\dfrac{12}{(2x+1)^3} & \text{if } x<0 \text{ and } x\neq -\dfrac{1}{2} \\ -\dfrac{4x}{(1+x^2)^2} & \text{if } 0<x<1 \end{cases}.$$

At 0, we have

$$f'_-(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x}=\lim_{x\to 0^-}\frac{g(x)-h(0)}{x}=\lim_{x\to 0^-}\frac{g(x)-g(0)}{x}=g'(0)=2$$

$$f'_+(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x}=\lim_{x\to 0^+}\frac{h(x)-h(0)}{x}=h'(0)=2,$$

so $f'(0)=2$. However,

$$f''_-(0)=\lim_{x\to 0^-}\frac{f'(x)-f'(0)}{x}=\lim_{x\to 0^-}\frac{\left[\dfrac{3}{(2x+1)^2}-1\right]-2}{x}=\lim_{x\to 0^-}\frac{-12(x+1)}{(2x+1)^2}=-12$$

$$f''_+(0)=\lim_{x\to 0^+}\frac{f'(x)-f'(0)}{x}=\lim_{x\to 0^+}\frac{\dfrac{2}{1+x^2}-2}{x}=\lim_{x\to 0^+}\frac{-2x}{1+x^2}=0,$$

so $f''(0)$ does not exist.

So we have

$$f'(x)=\begin{cases} \dfrac{3}{(2x+1)^2}-1 & \text{if } x<0 \text{ and } x\neq -\dfrac{1}{2} \\ \dfrac{2}{1+x^2} & \text{if } 0\leq x<1 \end{cases} \quad\text{and}\quad f''(x)=\begin{cases} -\dfrac{12}{(2x+1)^3} & \text{if } x<0 \text{ and } x\neq -\dfrac{1}{2} \\ -\dfrac{4x}{(1+x^2)^2} & \text{if } 0<x<1 \end{cases}.$$

Now $f'(x)=0$ only if $x=-\frac{1}{2}-\frac{\sqrt{3}}{2}$, and $f''(x)$ is always non-zero. Thus we have the following table.

x	$\left(-\infty,-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)$	$-\frac{1}{2}-\frac{\sqrt{3}}{2}$	$\left(-\frac{1}{2}-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$	$-\frac{1}{2}$	$\left(-\frac{1}{2},0\right)$	0	$(0,1)$
$f'(x)$	$-$	0	$+$	undef.	$+$	$+$	$+$
$f''(x)$	$+$	$+$	$+$	undef.	$-$	undef.	$-$
$f(x)$	↘	min.	↗	undef.	↗	?	↗

f has a local minimum point $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2},2+\sqrt{3}\right)$, no local maximum points and no points of inflection.

Now we look for asymptotes of the graph of f.

(i) It is clear that the line $x=-\frac{1}{2}$ is a vertical asymptote of the graph of f. On the other hand, the limit

$$\lim_{x\to 1^-}f(x)=\lim_{x\to 1^-}\arcsin\frac{2x}{1+x^2}=\frac{\pi}{2}$$

exists as a finite real number, so the line $x=1$ is not a vertical asymptote of the graph of f.

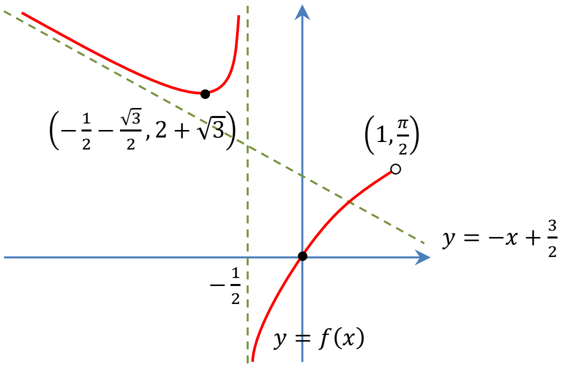
(ii) Since

$$\lim_{x\to-\infty}[f(x)-(-x)]=\lim_{x\to-\infty}\left[\frac{2x(1-x)}{2x+1}+x\right]=\lim_{x\to-\infty}\frac{3x}{2x+1}=\lim_{x\to-\infty}\frac{3}{2+\frac{1}{x}}=\frac{3}{2},$$

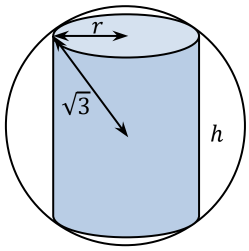
it follows that the line $y=-x+\frac{3}{2}$ is a slant asymptote of the graph of f.

Clearly, the graph of f has the only x- and y-intercept at the origin $(0,0)$.

The following shows a sketch of the graph of f.



11. Let the height, the base radius and the volume of the circular cylinder be h units, r units and V cubic units respectively. We have



$$\begin{cases} r^2+(h/2)^2=(\sqrt{3})^2 \\ V=\pi r^2 h \end{cases}.$$

The first equation gives $r^2=3-\frac{h^2}{4}$, so we obtain the objective function

$$V(h)=\pi\left(3-\frac{h^2}{4}\right)h=3\pi h-\frac{\pi h^3}{4}, \qquad \text{where } 0\le h\le 2\sqrt{3}.$$

Differentiating both sides we get

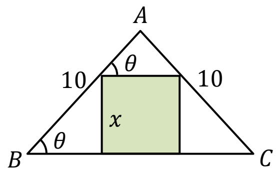
$$V'(h)=3\pi-\frac{3\pi}{4}h^2=\frac{3\pi}{4}(2-h)(2+h)$$

for every $0<h<2\sqrt{3}$. Now $V'(h)=0$ only if $h=2$.

- Since $V'(h)\begin{cases} >0 & \text{if } 0<h<2 \\ <0 & \text{if } 2<h<2\sqrt{3} \end{cases}$, $V(h)$ attains local maximum when $h=2$. This local maximum is also the global maximum of $V(h)$ on $[0,2\sqrt{3}]$.
- Alternatively, since $V''(2)=-\frac{3\pi}{2}(2)=-3\pi<0$, $V(h)$ attains local maximum when $h=2$. This local maximum is also the global maximum of $V(h)$ on $[0,2\sqrt{3}]$.
- Alternatively, since $V(h)$ is continuous on the closed interval $[0,2\sqrt{3}]$ with $V(0)=0$, $V(2)=4\pi$ and $V(2\sqrt{3})=0$, $V(h)$ attains global maximum when $h=2$.

Thus the volume of the circular cylinder is maximized when the height is 2 units and the base radius is $\sqrt{2}$ units.

12. Let the size of $\angle ABC$ be θ radians and the area and the length of a side of the square be A and x respectively. Then we have



$$\begin{cases} x\csc\theta + \dfrac{x}{2}\sec\theta = 10 \\ A = x^2 \end{cases}.$$

From the first equation we get the relation

$$x = \frac{10}{\csc\theta + \frac{1}{2}\sec\theta} = \frac{20}{2\csc\theta + \sec\theta}, \qquad \text{where } 0 < \theta < \frac{\pi}{2}.$$

It is clear that A is maximized if and only if x is maximized, so we just differentiate both sides of the above equality to get

$$\frac{dx}{d\theta} = \frac{-20(-2\csc\theta\cot\theta + \sec\theta\tan\theta)}{(2\csc\theta + \sec\theta)^2} = \frac{20(2\cos^3\theta - \sin^3\theta)}{(2\cos\theta + \sin\theta)^2}$$

for every $0 < \theta < \frac{\pi}{2}$. Now $\frac{dx}{d\theta} = 0$ only if $\tan^3\theta = 2$, which happens only if $\theta = \arctan\sqrt[3]{2}$. Since

$$\frac{dx}{d\theta}\begin{cases} > 0 & \text{if } 0 < \theta < \arctan\sqrt[3]{2} \\ < 0 & \text{if } \arctan\sqrt[3]{2} < r < \dfrac{\pi}{2} \end{cases},$$

x attains local maximum when $\theta = \arctan\sqrt[3]{2}$. This local maximum is also the global maximum of x (and of A) on $\left(0, \frac{\pi}{2}\right)$. Therefore the area of the square is maximized when the size of $\angle ABC$ is $\arctan\sqrt[3]{2}$ radians.

13. (a) Denote the length of BD as x m. By similar triangles we have $\frac{s+x}{8} = \frac{x}{\sqrt{1+x^2}}$, so the objective function is

$$s(x) = \frac{8x}{\sqrt{1+x^2}} - x, \qquad \text{where } 0 \le x \le 3\sqrt{7}.$$

Differentiating both sides, we get

$$s'(x) = \frac{8\sqrt{1+x^2} - (8x)\dfrac{1}{2\sqrt{1+x^2}}(2x)}{1+x^2} - 1 = \frac{8}{(1+x^2)^{3/2}} - 1$$

for every $0 < x < 3\sqrt{7}$. We have $s'(x) = 0$ only if $x = \sqrt{3}$.

⊙ Since

$$s''(x) = -\frac{24x}{(1+x^2)^{\frac{5}{2}}},$$

we have $s''(\sqrt{3}) = -\frac{3}{4}\sqrt{3} < 0$, and so $s(x)$ attains local maximum when $x = \sqrt{3}$. This local maximum of s is also the global maximum of s on $[0, 3\sqrt{7}]$.

⊙ Alternatively, since $s'(x)\begin{cases} > 0 & \text{if } 0 < x < \sqrt{3} \\ < 0 & \text{if } \sqrt{3} < x < 3\sqrt{7} \end{cases}$, $s(x)$ attains local maximum when $x = \sqrt{3}$. This local maximum of s is also the global maximum of s on $[0, 3\sqrt{7}]$.

⊙ Alternatively, since $s(x)$ is a continuous function on the closed interval $[0, 3\sqrt{7}]$ with $s(0) = 0$, $s(\sqrt{3}) = 3\sqrt{3}$ and $s(3\sqrt{7}) = 0$, $s(x)$ attains global maximum when $x = \sqrt{3}$.

The maximum value of s is therefore

$$s_{\max} = s(\sqrt{3}) = \frac{8\sqrt{3}}{\sqrt{1+3}} - \sqrt{3} = 3\sqrt{3}.$$

(b) We have

$$p(x) = \frac{1}{2}\left(1+\sqrt{8^2-(s+x)^2}\right)s = \frac{1}{2}\left(1+\frac{8}{\sqrt{1+x^2}}\right)\left(\frac{8x}{\sqrt{1+x^2}}-x\right)$$

$$= \frac{32x}{1+x^2} - \frac{x}{2}, \qquad \text{where } 0 \le x \le 3\sqrt{7}.$$

Differentiating both sides, we have

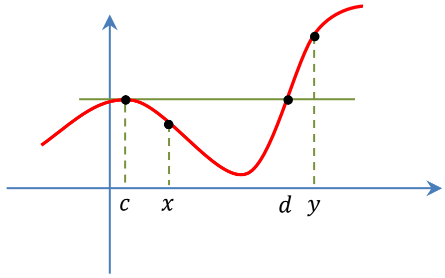
$$p'(x) = \frac{32(1+x^2)-(32x)(2x)}{(1+x^2)^2} - \frac{1}{2} = \frac{32(1-x^2)}{(1+x^2)^2} - \frac{1}{2}$$

for every $0 < \sqrt{x} < 3\sqrt{7}$. Since

$$p'(\sqrt{3}) = \frac{32(1-3)}{(1+3)^2} - \frac{1}{2} = -\frac{9}{2} \neq 0,$$

it follows that $\sqrt{3}$ is not a critical number of p. Therefore when s attains a maximum, i.e. when $x=\sqrt{3}$, p does not attain a maximum.

14. Suppose that f does not attain global maximum at c. Then $f(y) > f(c)$ for some $y \in I$, and without loss of generality we may assume that $y > c$. Now since f attains local maximum at c, if we pick a number $x \in (c, y)$ which is near c, then we have $f(x) \le f(c)$. Applying the Intermediate Value Theorem to the continuous function f, we see that there exists $d \in [x, y]$ such that $f(d) = f(c)$.



Now since $c < d$, f is continuous on $[c, d]$ and differentiable on (c, d), and $f(c) = f(d)$, according to **Rolle's Theorem** there exists $t \in (c, d)$ such that $f'(t) = 0$, i.e. t is another critical number of f apart from c. This gives a contradiction, so f must attain global maximum at c. ■